# Decoherence bounds on the capabilities of cold trapped ion quantum computers

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**ABSTRACT**

Using simple physical arguments we investigate the capabilities of a quantum computer based on cold trapped ions of the type recently proposed by Cirac and Zoller. From the limitations imposed on such a device by decoherence due to spontaneous decay, laser phase coherence times, ion heating and other possible sources of error, we derive bounds on the number of laser interactions and on the number of ions that may be used. As a quantitative measure of the possible performance of these devices, the largest number which may be factored using Shor's quantum factoring algorithm is determined for a variety of species of ion.

**Keywords**: quantum computing, ion traps, decoherence

## 1. INTRODUCTION

A quantum computer stores binary numbers in the quantum states of two-level systems ("qubits"), allowing the possibility of computation with coherent superpositions of numbers[1]. Because a single quantum operation can affect a superposition of many numbers in parallel, a quantum computer can efficiently solve certain classes of problems that are currently intractable on classical computers, such as the determination of the prime factors of large integers[2]. These problems are of such importance that there is now considerable interest in the practical implementation of a quantum computer[3,4]. There are three criteria which designs for quantum computers must meet: the qubits must be sufficiently isolated from the environment so that the coherence of the quantum states can be maintained throughout the computation; there must be a method of controlling the states of the qubits in order to effect the logical "gate" operations; and there must be a highly efficient method for measuring the final quantum state in order to find the answer.

J. I. Cirac and P. Zoller of the University of Innsbruck have proposed what seems to be the most promising design for the implementation of a quantum computer to date[5]. A number of identical ions are trapped and cooled in a linear radio-frequency quadrupole trap to form a quantum register. The radio-frequency trap potential gives strong confinement of the ions in the Y and Z directions transverse to the trap axis, while an electrostatic potential forces the ions to oscillate in an effective harmonic potential in the axial direction (X) (see fig.1). After laser cooling the ions become localized along the trap axis with a spacing determined by their Coulomb repulsion and the confining axial potential. The normal mode of the ions' collective oscillations which has the lowest frequency is the axial center of mass (CM) mode, in which all the trapped ions oscillate together. A qubit is the electronic ground state $|g\rangle$ and a long-lived excited state $|e\rangle$ of the trapped ions. The electronic configuration of individual ions, and the quantum state of their collective CM vibrations can be manipulated by coherent interactions of the ion with a laser beam, in a standing wave configuration, which can be pointed at any of the ions. The CM mode of axial vibrations may then be used as a "quantum data bus" to implement the quantum logical gates. Once the quantum computation has been completed, the readout is performed through the mechanism of quantum

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jumps. Several features of this scheme have been demonstrated experimentally, mostly using a *single* trapped ion[4,6]. More detailed discussions of the Cirac and Zoller design have been given by Steane[7] and by James[8].

The unavoidable interaction of a quantum computer with its environment places considerable limitations on the capabilities of such devices[9]. In this paper we present a quantitative assessment of these limitations for a computer based on the Cirac-Zoller cold-trapped-ion design[10]. There are two fundamentally different types of decoherence during a computation: the intrinsic limitation imposed by spontaneous decay from various quantum states of the ions; and practical limitations such as the random phase fluctuations of the laser driving the computational transitions or the heating of the ions' vibrational motion. One could, in principle, expect that as experimental techniques are refined, the effects of these practical limitations may be reduced until the intrinsic limit of computational capability due to spontaneous emission is attained.

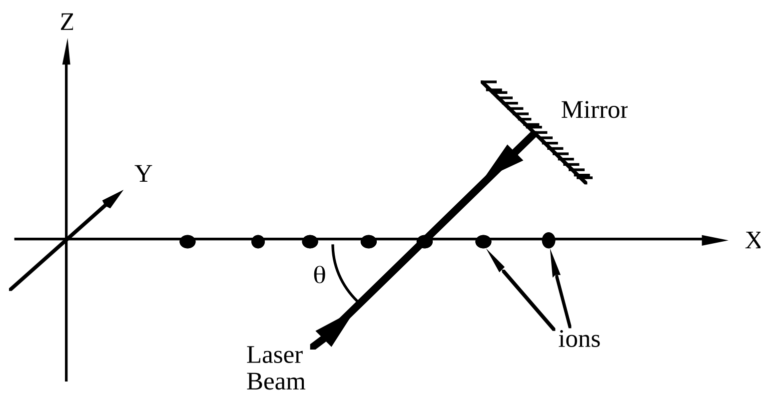


Figure 1. A schematic illustration of the Cirac-Zoller quantum computer. The laser beam is in a standing wave configuration and can be steered from ion to ion.

## 2. FUNDAMENTAL PERFORMANCE CONSTRAINTS

### 2.1 Effect of extraneous phonon states

There are two types of laser pulse that are required in order to realize Cirac and Zoller's scheme for quantum computation. The first are pulses that are tuned precisely to the resonance frequency of the |e> to |g> transition of the qubits, ideally configured so that the ion lies at the *node* of the laser standing wave ("V-pulses"); the second type of pulse is tuned to the CM phonon sideband of the transition, arranged so that the ion lies at the *antinode* of the standing wave ("U-pulses")[11]. It is the second type of pulse, which can excite both the internal degrees of freedom of the ion *and* the motion of the ions in the trap, which is the most challenging experimentally, and it is the ability to execute successfully these pulses that is an important limiting factor in the realization of a practical device. The Hamiltonian for the interaction of the U-pulses is given by the following expression[5]:

$$\hat{H} = \frac{\hbar\eta}{2\sqrt{L}}\Omega\Big[\left|e\right\rangle\left\langle g\right|\hat{a}e^{-i\phi} + \left|g\right\rangle\left\langle e\right|\hat{a}^{\dagger}e^{i\phi}\Big] \ . \tag{1}$$

In this formula, $\Omega$ is the Rabi frequency for the laser-ion interaction, $L$ is the number of ions in the trap, $\hat{a}$ ($\hat{a}^{\dagger}$) is the annihilation (creation) operator for phonons of the CM mode and $\eta = (\hbar\omega^2\cos^2\theta/2Mc^2\nu_x)^{1/2}$ is the Lamb-Dicke parameter (here $\omega$ is the laser angular frequency, $\theta$ the angle between the laser and the trap axis, $\nu_x$ is the angular frequency of the ions' axial CM mode and $M$ the mass of each ion). A careful calculation[8], based on a perturbative analysis of the excitation of phonon modes other than the CM mode, shows that this Hamiltonian is valid provided that $(2.6\Omega\eta/\nu_x\sqrt{L})^2 << 1$ . The duration of each $2\pi$ U-pulse is $t_U=2\pi\sqrt{L}/\Omega\eta$. For simplicity we will assume that *all* of the U-pulses required for the calculation are of this duration. In order to avoid excitation of extraneous phonon modes, the duration of each U-pulse must be limited by the following inequality:

$$t_U > \frac{16.3}{\nu_x}, \tag{2}$$

where the factor of 16.3 comes from $2\pi \times 2.6$. This result can also be obtained approximately from the simple uncertainty principle argument that there must not be appreciable power at the frequencies of the resonances associated with other lattice vibrations.

**2.2 Effect of spontaneous emission from the upper level of the qubit**

The influence of spontaneous emission on a quantum computation with trapped ions depends on the natural lifetime of the excited state |e> of each qubit; the number of ions, $L$, being used; and the quantum states of those ions. The number of ions which are not in their ground states varies as the calculation progresses, with ancillary ions being introduced and removed from the computation. The progression of the ions' states can be characterized well by an effective number of ions, $L'$, which have a non-zero population in the excited state |e>. In the case of Shor's factoring algorithm[2], a reasonable estimate is $L' \approx 2L/3$.

To estimate the effect of decoherence during the implementation of Shor's algorithm, we will consider the following simple process: a series of laser pulses of appropriate strength and duration ($\pi/2$ pulses) is applied to $2L/3$ ions, causing each of them to be excited into an equal superposition state (|e>+|g>)/√2. After an interval $T$, a second series of laser pulses ($-\pi/2$ pulses) is applied, which, had there been no spontaneous emission, would cause each ion to be returned to its ground state. This is the "correct" result of our pseudo-computation. If there were spontaneous emission from one or more of the ions, then the ions would finish in some other, "incorrect" state. This process involves the sort of superposition states that will occur during a typical quantum computation, and so the analysis of decoherence effects in this procedure will give some insight into how such effects influence a real computation. The probability of obtaining a correct result is $P(T) \approx 1 - LT/6\tau_0$, where $\tau_0$ is the natural lifetime of the excited state |e>. Thus the effective coherence time of the computer is $6\tau_0/L$.

The total time taken to complete a calculation will be approximately equal to the number of laser pulses required multiplied by the duration of each pulse. The time taken to switch the laser beam from ion to ion is assumed to be negligible. The interaction of U-pulses with the ions is considerably weaker than the V-pulses, and so, assuming constant laser intensity, the U-pulse duration must be longer. Hence, in calculating the total time required to perform a quantum computation, we will neglect the time required for the V-pulses. Because the entire calculation must be performed in a time less than the coherence time of the computer, we obtain the inequality $N_U t_U < 6\tau_0/L$. If we substitute from (2) we obtain the following constraint on the values of $N_U$ and $L$:

$$N_U L < 0.37 \tau_0 \nu_x \quad , \tag{3}$$

where $N_U$ is the total number of U-pulses required for the calculation.

**2.3 Effect of extraneous atomic states**

Figure 2 shows a simplified energy level diagram for typical alkali-like ions which are suitable for use in a quantum computer of the type we are discussing. A laser field, precisely tuned to the |e> to |g> transition wavelength is used to perform Rabi flips between these two levels. However as these operations are being performed there will be a small probability of erroneously exciting other *short lived* quantum levels of the ion; if one of these levels were spontaneously to emit a photon during the computation, then the coherence of the computer would be lost.

If the average probability of some extraneous level |3> being excited is $P_3$, then to avoid decoherence due to this mechanism we require that $N_U t_U P_3/\tau_3 < 1$, where $\tau_3$ is the lifetime of the extraneous level |3>. The probability $P_3$ can be shown to be given approximately by the formula $P_3 \approx \Omega^2 \tau_0 \lambda^3_3 / 4\tau_3 \Delta^2 \lambda^3$ , where $\Delta$ is the detuning between the transition |e> to |g> and the transition |3> to |g>, $\lambda$ is the resonance wavelength of the |e> to |g> qubit transition and $\lambda_3$ is the resonance wavelength of the |3> to |g> transition. Using the formula for $t_U$, (2), and the definition of the Lamb-Dicke parameter, we therefore obtain the following constraint:

$$N_U L < \frac{32.6\tau_3^2 \Delta^2 \text{h}(\cos\theta)^2 \lambda}{\tau_0 M \lambda_3^3 \nu_x^2} \quad . \tag{4}$$

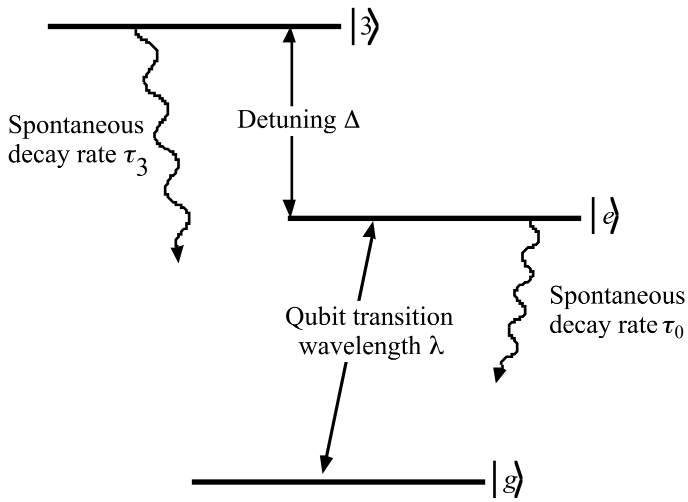


Figure 2. Simplified energy level diagram for alkali like ions. The state |g> is the ground state, |e> is the metastable first excited state used as the upper level of the qubit and |3> is some short-lived "extraneous" level.

**2.4 Effect of laser spot size**

In order to attain the highest possible computational capability, one will need to minimize the duration of each laser pulse. Hence, according to (2), it will be advantageous to employ an ion trap with the largest possible value of the trap frequency $\nu_x$. However, the axial frequency cannot be made arbitrarily large because, in order to avoid crosstalk between adjacent ions, the minimum inter-ion spacing must be much larger than the size of the focal spot of the laser beam. The minimum separation distance between two ions occurs at the center of the string of ions, which can be calculated by solving for the equilibrium positions of the ions numerically, resulting in the following expression[8] :

$$x_{\min} \cong \left( \frac{e^2}{4\pi\varepsilon_0 \nu_x^2 M} \right)^{1/3} \frac{2.0}{L^{0.56}} \quad , \tag{5}$$

where e is the electron charge, $\varepsilon_0$ is the permittivity of a vacuum and M is the mass of each ion. The spatial distribution of light in focal regions is well known[12]. The approximate diameter of the focal spot is $x_{spot} \approx \lambda F$, where $\lambda$ is the laser wavelength and F the focal ratio of the focusing system (i.e. the ratio of the focal length to the diameter of the exit pupil). Hence the requirement that the ion separation must be large enough to avoid cross-talk between ions, i.e. that $x_{min} >> x_{spot}$, leads to the following constraint on the value of the trap frequency:

$$\nu_x L^{0.84} << \sqrt{\frac{6.1 \; e^2}{4\pi\varepsilon_0 M \lambda^3 F^3}} \quad . \tag{6}$$

## 3. PERFORMANCE CONSTRAINTS BASED ON ATOMIC DATA

It will be convenient to write the trap angular frequency $\nu_x$ in terms a frequency in units of MHz, i.e., $\nu_x = 2\pi f \times 10^6$. Then the inequalities (3) and (4) may then be written in the simplified form

$$N_U L < Af \,, \quad N_U L < B \big/ f^2 \tag{7}$$

where the constants $A$ and $B$ depend on the species of ion chosen (we have assumed and angle $\theta = 80^o$ to evaluate the parameter $B$). We will be considering four different species of ion, all of which have the property that their first excited state above the ground state is metastable. These ions are:

(i) $Hg^+$: mass number 198; |e> is a sublevel of the $5d^9 6s^2\ ^2D_{5/2}$ level, |g> is a sublevel of the $5d^{10}6s\ ^2S_{1/2}$ level and |3> is a sublevel of the $5d^{10}6p\ ^2P_{1/2}$ level: $\lambda$ = 281.5 nm; $\tau_0$ = 0.098 sec, $\lambda_3$= 194.2 nm; $\tau_3$= 2.3 nsec.

(ii) $Ca^+$: mass number 40; |e> is a sublevel of the 3 $^2D_{5/2}$ level, |g> is a sublevel of the 4 $^2S_{1/2}$ level and |3> is a sublevel of the 4 $^2P_{1/2}$ level: $\lambda$ = 732 nm; $\tau_0$ = 1.16 sec, $\lambda_3$= 397 nm; $\tau_3$= 7.7 nsec.

(iii) $Ba^+$: mass number 138; |e> is a sublevel of the 5 $^2D_{5/2}$ level, |g> is a sublevel of the 6 $^2S_{1/2}$ level and |3> is a sublevel of the 6 $^2P_{1/2}$ level: $\lambda$ = 1.761 μm; $\tau_0$ = 47 sec, $\lambda_3$= 493 nm; $\tau_3$= 11 nsec.

(iv) $Sr^+$: mass number 88; |e> is a sublevel of the 4d $^2D_{5/2}$ level, |g> is a sublevel of the 5s $^2S_{1/2}$ level and |3> is a sublevel of the 5p $^2P_{1/2}$ level: $\lambda$ = 687 nm; $\tau_0$ = 395 msec, $\lambda_3$= 422 nm; $\tau_3$= 7.9 nsec.

References for this data are given in[8]. Values of the parameters $A$ and $B$ for these ions are given in table1.

| Ion | $A$ ($\times 10^6$) | $B$ ($\times 10^6$) | $f_0$(MHz) | $L_{max}$ |
|---|---|---|---|---|
| $Hg^+$ | 0.228 | 0.150 | 0.870 | 222 |
| $Ca^+$ | 2.70 | 0.112 | 0.346 | 314 |
| $Ba^+$ | 109 | 0.00327 | 0.0310 | 544 |
| $Sr^+$ | 0.918 | 0.0768 | 0.437 | 166 |

Table 1: Values of important parameters defined in the text for four different types of ion.

The two inequalities given in (7) imply that there is a optimum value of the trap frequency at which the product $N_U L$ will have a maximum allowed value. This optimum frequency is given by the following formula:

$$f_0 = \left(B/A\right)^{1/3}. \tag{8}$$

The values of $f_0$ for our sample ions are given in table 1. Thus the constraint on the performance of the quantum computer due to decoherence now reads

$$N_U L < A f_0. \tag{9}$$

This relationship is plotted in figure 3 for the four different species of ion we are considering.

When operating the ion trap computer at frequency $f_0$, (6) implies that there is a maximum value for the number of ions $L_{max}$ which can be used in the trap. If more ions than $L_{max}$ ions are loaded, then the laser beam will be unable to resolve the individual qubits, resulting is errors in the calculation. The values of $L_{max}$ can be calculated from (6); they are also given in table 1 (we have assumed an angle $\theta = 80^o$ and a focal ratio $F$ = 1). As can be seen, when operating the trap with a few dozen ions at the optimum frequency given by (8), there should be no particular difficulty about resolving the ions.

## 4. QUANTUM ALGORITHMS

We will now apply the bound (9) to Shor's factor finding algorithm[2]. Let $l$ be the number of bits of the integer we wish to factor. An analysis of one version of this algorithm[13] shows that the required number of ions and U-pulses are approximately given by:

$$L = 5l + 4 \quad , \tag{10}$$

$$N_U \approx [292l^3 - 151l^2 + 8l + 2]/3 \quad . \tag{11}$$

Equations (10) and (11) define a curve in ($L$, $N_U$) space, which taken in conjunction with the inequality (9) allows us to determine the largest number of ions that can be used to implement the simple version of Shor's algorithm (without the use of quantum error correction) in an ion trap computer with bounded loss of coherence. The linear relationship between $L$ and $l$, (10), can then be used to determine the largest number that can be factored .

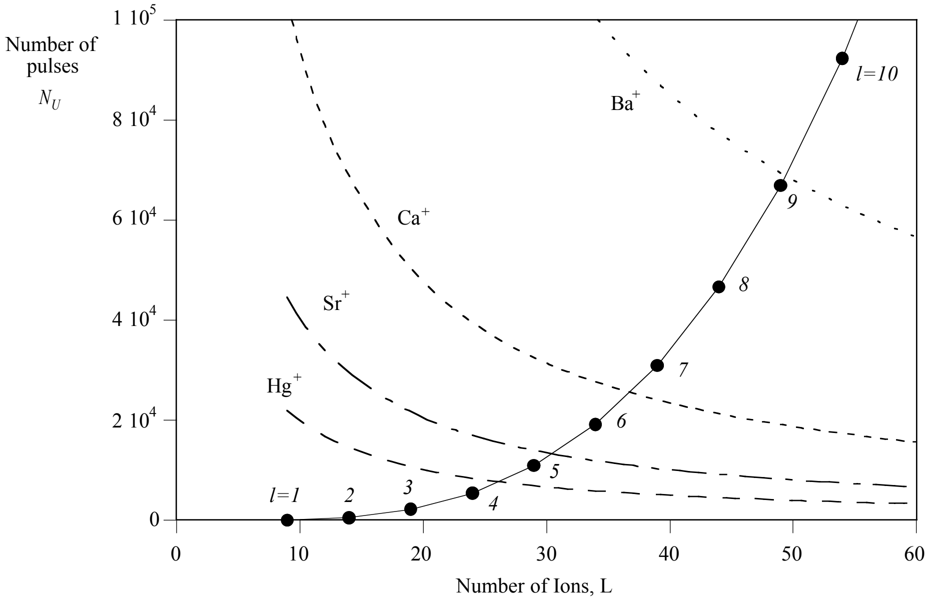


Figure 3. The bounds on the numbers of ions, $L$, and the number of U-pulses, $N_U$, that may be used in a quantum computation without loss of coherence. The allowed values of $N_U$ and $L$ lie to the left of the curves. Curves for four ions are plotted. The unbroken line is the "factorization curve", specified by (10) and (11), which represents those values of $N_U$ and $L$ which are required for execution of Shor's algorithm; the heavy black dots on this line represent the values of $N_U$ and $L$ required to factor a number of $l$ bits ($l$ = 1, 2, ...10).

In figure 3 we have plotted the curves which limit the allowed values of L and $N_U$, as given by (9). We have also plotted, with a solid line, the "curve of factorization" defined by (10) and (11). The intersection of the limiting curves for the different ions with the curve of factorization gives us an estimate of the largest number that can be factored; for a Cirac-Zoller quantum computer based on $Hg^+$, $Ca^+$, $Ba^+$ or $Sr^+$ ions the largest numbers that can be factored are 4 bits, 6 bits, 9 bits and 5 bits, respectively. It should be remembered that these results are only estimates of what can be done before spontaneous emission starts to become a problem; larger quantum calculations could be tackled if one were to accept a higher probability of error or to adopt some scheme of quantum error correction. Although these results may seem small, they nevertheless represent a large number of quantum logic operations (for example, to factor a 6 bit number requires of the order of $2 \times 10^4$ laser operations). Thus our results suggest that a reasonable degree of optimism is justified regarding the possibility of performing extended quantum logic operations using ion trap quantum computers.

## 5. EXPERIMENTAL DECOHERENCE EFFECTS

One may calculate the limits on factoring due to other causes of decoherence by a similar procedure to that used above. In this case, we will assume that the loss of quantum coherence due to sundry effects such as random fluctuations of the laser phase or the heating of the ions' vibrational motion can be characterized by a single coherence time $\tau_e$. The effects of other causes of error, such as imprecise measurement of the areas of $\pi$-pulses, which do not result in decoherence but nevertheless lead to incorrect results in a computation, can also be characterized by the time $\tau_e$. Thus, in place of (3) we now have the inequality $N_U t_U < \tau_e$. Using (2) we obtain the following constraint on the value of the number of laser pulses $N_U$ which can be used in a quantum computation without significant loss of coherence:

$$N_U < 3.85 \times 10^5 f_0 \tau_e \ . \qquad (12)$$

Using (11) and the values of $f_0$ given in table 1, one can solve (12) to determine the number of bits $l$ in the largest number which may be factored. In this case the value of $l$ will depend on the specie of ion and the value of the coherence time $\tau_e$. In figure 4 we have plotted the values of $l$ as a function of the experimental coherence time for the four species of ions discussed above. As $\tau_e$ increases, the largest number that can be factored also increases, until the limit due to spontaneous emission discussed above is attained.

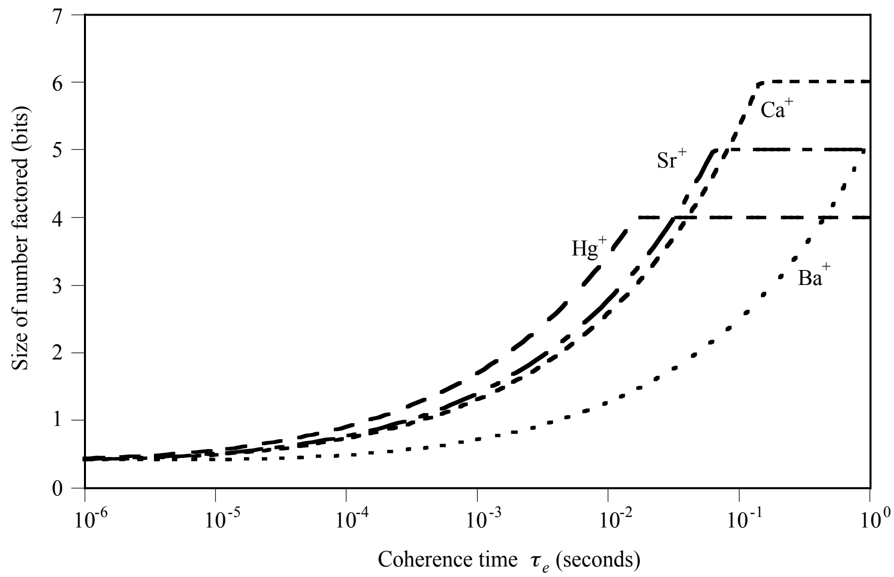


Figure. 4. The variation of the number of bits $l$ in the largest integer that may be factored with the experimental coherence time for the ions discussed in the text. The plateaus in the curves for $Hg^+$, $Ca^+$ and $Sr^+$ are the limits determined by spontaneous emission discussed above.

The slowest heating rate for a single trapped ion so far reported is 6 phonons per second (i.e. $\tau_e = 0.17$ sec)[14], and the laser phase coherence times longer than $10^{-3}$ sec have been achieved by several groups[15]. Comparing these numbers with fig.4, we see that, in principle, current technology is capable of producing a quantum computer that could factor at least small numbers of several bits.

The various causes of experimental decoherence which are mentioned above are all under investigation. It is not clear, for example, how laser phase fluctuations will affect quantum computations; it may be the case that the laser need be coherent only over the period required to execute each quantum gate operation. Furthermore, the heating rate of the ions' vibrational motion as a function of the number of trapped ions is not known. Other methods of coherent population transfer, which may be less susceptible to the effects of phase fluctuations, for example stimulated Raman transitions between magnetic sublevels of the ground state may offer considerable advantages.

We have ignored in the above calculation the influence of quantum error correction in the calculation. It is clear that if quantum computation is to overcome decoherence and other errors, then some form of error correction must be used extensively. This is a field that is the subject of considerable ongoing research. The latest results suggest that if quantum gate operations can be performed within some threshold degree of accuracy, estimated to be below $10^{-6}$, then arbitrarily complex quantum computations can be performed reliably[16]. These theoretical results give a challenging but not necessarily impossible goal for various technologies to aim at. The results presented here give reasonable grounds for optimism: for example, to factor a 6 bit number, (which should be possible using a quantum computer based on $Ca^+$ ions) requires of the order of $2.0 \times 10^4$ operations. Thus taking into account decoherence effects, the degree of accuracy of each operation will be of the order of $5.0 \times 10^{-5}$, which is encouragingly near the required accuracy threshold. Note however we have not taken into account errors due to operational causes, such as inexact pulse areas or laser intensity fluctuations.

## 6. CONCLUSIONS

We have derived quantitative bounds which show how the computational capabilities of a trapped ion quantum computer depend on the relevant physical parameters and determine the computational “space” ($L$) and “time” ($N_U$) combination that should be optimized for the most effective algorithms. The effect of this bound has been illustrated by calculating the size of the largest number that may be factored using a computer based on various species of ion. Our results show there is reason for cautious optimism about the possibility of factoring at least small numbers using a first generation quantum computer design based on cold trapped ions. However, the large number of precise laser operations required and the number of ions involved indicates that even this computationally modest goal will be extremely challenging experimentally.


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